or

$$P\pi r^2\Big[(1+2\,H/r)+\frac{U(0)+u(0)}{r}+\frac{1}{rP}\int\limits_0^l p\left(\frac{dU}{dx}+\frac{du}{dx}\right)dx\Big]\,.$$

The formula for the effective area is now

$$A_P=\pi r^2\left[(1+2\,H/r)+\frac{U(0)+u(0)}{r}+\right.$$
$$\left.+\frac{1}{rP}\int\limits_0^l p\left(\frac{dU}{dx}+\frac{du}{dx}\right)dx\right] \qquad (2.2)$$

of which expression (2.1) is a special case with U and u zero. We may evidently obtain (2.2) more directly by visualising the neutral surface as the effective boundary of the piston in which case the frictional force corresponding to b) above vanishes and we are left simply with the pressure forces acting on the effective boundaries of the piston. We also have the equivalent form

$$A_P=\pi r^2\left[1+2\,u(0)/r+\right.$$
$$\left.+\frac{2}{rP}\left(\int\limits_0^l -h\,\frac{dp}{dx}\cdot dx+\int\limits_0^l p\,\frac{du}{dx}\cdot dx\right)\right] \qquad (2.3)$$

which is convenient for use when the integral $\int\limits_0^l -h\,\frac{dp}{dx}\,dx\,\left(or\int\limits_0^P h\,dp\right)$ is of interest, as is the case, for example, when the flow method (section 5) is considered.

b) The effects of special assumptions

The problem of calculating the actual changes of effective area of practical designs of piston-cylinder assembly, on the basis of the above general formulae, is complicated. It would be necessary to know the interrelated quantities u, U and p as functions of x, and since the pressure gradient dp/dx is governed by the normal equation of viscous flow (see equation 5.1), the pressure dependence of the coefficient of viscosity would also need to be taken into account. It is not, however, the aim of the present paper to attempt such calculations, but rather to describe direct experimental methods for the accurate determination of the distortion factors with the minimum of assumptions regarding the detailed behaviour of the system. We therefore consider only certain special cases which are useful in the applications which follow.

A useful approximation may be derived from the foregoing equations by assuming that the component of $u(x)$ or $U(x)$ due to the fluid pressure in the interspace between piston and cylinder may be taken to be proportional to the pressure $p(x)$ *at the same position*. The relevant terms in the integrals on the right hand side then become integrable without the necessity for any further knowledge of the actual functional forms of $u(x)$, $U(x)$ or $p(x)$. There is fair support from elastic theory for this assumption, more especially in the case of the solid cylinder in which the length is large compared with its radius, a condition which applies to the pistons of most pressure balance assemblies other than those catering for only a low range of pressure. Chree (1889, 1901) has given polynomial solutions for the equilibrium of a finite solid cylinder for cases in which the lateral pressure is either a linear or quadratic function of the axial co-ordinate. The conditions are satisfied by functions $u(x)$ and $p(x)$ which are accurately proportional, provided the normal tractions over the flat ends, instead of being identically zero, are assumed only to average to zero. By Saint-Venant's principle, however, the effect of this disturbance will be appreciable for only a short distance from each end, and may be neglected if the ratio of length to radius is considerable. The constant of proportionality is the same as in the case of uniform pressure on a solid cylinder of infinite length. Filon (1902) has obtained solutions for pressure distributions expressed in series of trigonometric functions of x which lead to a similar result provided the wavelengths involved are fairly large compared with the radius. The effects of discontinuous pressure distributions, or narrow bands of applied pressure, have also been discussed (Barton 1941; Rankin 1944; Tranter & Craggs 1947), with the general result that even the effects of discontinuities are largely lost at an axial distance of only about half the radius. If, therefore, the pressure changes along the length of the assembly are reasonably smooth, no great error is likely to be incurred by applying this assumption to the *piston* of the assembly. Taking into account the additional change of radius due to the end thrust on the piston, it is easily shown that the relevant terms involving u on the right hand side of equation (2.2) reduce to $P\,(3\,\sigma-1)/2\,E$ where E and σ are respectively Young's modulus and Poisson's ratio, so that we now have, using also (2.1),

$$A_P=A_0\left[1+\frac{P(3\,\sigma-1)}{2\,E}+\frac{U(0)}{r}+\frac{1}{rP}\int\limits_0^l p\,\frac{dU}{dx}\cdot dx\right]\,. \qquad (2.4)$$

Another useful form, obtained directly from (2.3), is

$$A_P=\pi r^2\left[1+\frac{P(3\,\sigma-1)}{E}+\frac{2}{rP}\int\limits_0^P h\,dp\right]\,. \qquad (2.5)$$

The application of a similar assumption to deal with the effects of internal pressure in a hollow cylinder with thick walls is less secure. Chree (1901) has given a corresponding solution with $U(x)$ and $p(x)$ proportional for the case where $p(x)$ is a linear function of x, but its validity would depend on the conditions assumed at the ends. The case of a discontinuous distribution of pressure has been considered briefly by Tranter (1946). In the ideal case of a cylinder whose length is large compared with its radius and wall thickness, where the working section is removed some distance from the points of attachment of the ends, and the pressure distribution is reasonably smooth, a useful approximation may result. Proceeding from equation (2.4), and taking for definiteness the case where the cylinder walls are not subjected to longitudinal stress, we then obtain (Love 1952), denoting by R' the outer radius of the cylinder,

$$A_P=A_0\left\{1+\underset{\text{(piston)}}{\frac{P}{2\,E}\,(3\,\sigma-1)}+\underset{\text{(cylinder)}}{\frac{P}{2\,E}\left[\frac{(1+\sigma)\,R'^2+(1-\sigma)\,R^2}{R'^2-R^2}\right]}\right\}$$

or, combining the distortion terms,

$$A_P = A_0\left[1 + \frac{P}{E}\left(2\,\sigma + \frac{R^2}{R'^2 - R^2}\right)\right] \ . \qquad (2.6)$$

In the limiting case with R'/R effectively infinite this reduces to the simple expression

$$A_P = A_0\left(1 + \frac{2\,\sigma}{E} \cdot P\right) \ . \qquad (2.7)$$

Equations (2.4) to (2.7) are a useful basis for the development of certain small correction terms which arise in the theory of the similarity and flow methods.

3. The Similarity Method

a) Principle of the method

In normal practice the assemblies for which calibrations are principally required are constructed of steel. The principle adopted in the similarity method is first to determine the ratio of the effective area of the steel piston-cylinder assembly of given type, at a series of pressures, to that of a precisely similar assembly constructed of a material having a substantially different elastic modulus. This procedure determines the difference between the distortion factors of the two assemblies as a function of pressure. A second relation — the quotient of the two distortion factors — is obtained from measurements of the elastic moduli of the two materials. The combination of these results then allows the distortion factor of each assembly to be derived, as a function of pressure, in absolute terms.

b) Ideal theory of the similarity method

In its ideal form the similarity method is extremely simple, and involves no assumption regarding the form of distortion of the assembly when under pressure. In the ideal situation the two materials are regarded as elastically isotropic, with linear stress-strain relationships and identical Poisson's ratios over the range of stress involved. The two assemblies are assumed to be constructed to the same principal dimensions and to have accurately straight and circular pistons and cylinder bores. Ideally, the initial radial separations between the components of the two assemblies should be in inverse ratio to their elastic moduli, although it is found in practice that this condition is not critical. These conditions ensure that, as the distortion changes with increasing pressure, the annular channels between piston and cylinder will remain similar in form and that consequently the pressure distributions along the lengths of the channels will always remain the same for the same total applied pressure.

If these assumptions are realised the distortion terms in the expressions for the effective areas will remain in a fixed numerical ratio as the pressure is varied. In other words the effective areas A_P and B_P of the two assemblies at the applied pressure P may be written in the form,

$$A_P = A_0[1 + \lambda_A f(P)]\,;\ B_P = B_0[1 + \lambda_B f(P)] \qquad (3.1)$$

where λ_A, λ_B are constants in inverse ratio to the elastic moduli, and $f\,(P)$ is a function of the applied pressure of which the form is unknown but is the same in both cases. Bearing in mind that the distortion terms are normally very small compared with unity, the ratio of the areas may be expressed in the form

$$\frac{A_P}{B_P} = \frac{A_0}{B_0}\,[1 + (\lambda_A - \lambda_B)f(P)] \qquad (3.2)$$

and writing $\lambda_B = k\lambda_A$, where k is a constant, we obtain

$$\frac{A_P}{B_P} = \frac{A_0}{B_0}\,[1 + (1 - k)\lambda_A f(\mathrm{P})] \ . \qquad (3.3)$$

The ratio A_P/B_P, and consequently the function $(1 - k)\ \lambda_A\, f\,(P)$, may be determined easily and with high precision by simply measuring the loads on the two pistons when the assemblies are balanced against one another and in equilibrium at the same pressure, and carrying out this procedure at a series of pressures over the appropriate range. The quotient, k, of the elastic moduli may be determined by the standard methods for the measurement of elastic constants. It is clear that in the ideal conditions postulated these two procedures suffice to establish the values of the distortion terms $\lambda_A\, f(P)$ and $\lambda_B\, f(P)$ to an accuracy limited only by the sensitivity of the balancing process and the precision to which the elastic constants are known. In general it is found to be the second factor which eventually limits the accuracy attainable, and to obtain the best precision k should evidently differ substantially from unity.

It is of particular interest that the rheological properties of the pressure transmitting fluid — e. g. dependence of coefficient of viscosity upon pressure — are entirely eliminated in the similarity procedure.

In order to simplify further discussion it is useful at this point to anticipate one practical result of the investigation, viz. that in most cases the distortion is very closely represented by a linear function of the applied pressure so that we may normally replace $f(P)$ by P, when the quantities λ_A and λ_B may be regarded simply as pressure coefficients having the dimensions $(\text{pressure})^{-1}$. Thus we may write instead of (3.1), $A_P = A_0\ (1 + \lambda_A P)$ etc., in all but exceptional cases.

c) Effect of departures from the ideal conditions

It would be a somewhat fortunate coincidence if the ideal assumptions were completely realised in a pair of actual metals having a sufficiently large ratio of elastic moduli, and also adequate tensile strengths, to justify their use in practice, and it is necessary to consider to what extent minor departures may be tolerated, or whether reliable correction terms can be developed. Materials showing appreciable elastic anisotropy are hardly worth consideration owing to the greatly increased complexity of the distortion of the system, and the labour of determining the complete set of elastic constants over a wide range of stress. Again, a pronounced departure from a linear stress-strain relation would introduce awkward complications; small departures may be tolerable, subject to a corresponding uncertainty in the value of the elastic modulus. In the case of a moderate difference in the values of the Poisson's ratios, however, it is not difficult to formulate a correction term. This is small and need only be evaluated approximately. For this purpose we make use of the formula (2.4), and express the distortion coefficients in the form $\lambda_A = \theta_A + \varphi_A \ldots$ where

$$\theta_A = (3\ \sigma_{(A)} - 1)/2\ E_{(A)}\ \ldots \qquad (3.4)$$

and φ_A is that part of λ_A which is explicitly dependent